The only remaining point we have to notice is that Mr. Galbraith adopts the innovation of defining $N_x$ as equal to $D_x + D_{x+1} + \ldots$ and yet speaks of this as "Davies's notation." This is introducing a fresh element of confusion into a question already too much involved. According to Davies's notation and formulæ

$$N_x = D_{x+1} + D_{x+2} + \ldots$$

and the use of Davies's symbol, $N_x$, to denote a different quantity, is a practice much to be deprecated, as very likely to lead to great confusion and error. All that is necessary to prevent mistakes is the use of a new symbol instead of Davies's; and we would suggest $\mathbb{N}_x$ for the purpose; so that we should have

$$N_x = D_{x+1} + D_{x+2} + \ldots$$
$$\mathbb{N}_x = D_x + D_{x+1} + D_{x+2} + \ldots$$

and
$$\mathbb{N}_x = N_x + D_x.$$

Then, $a_x$ denoting as usual an annuity on a life, $x$, if $\mathbf{a}_x$ denote an annuity-due, or an annuity payable in advance, on the same life, we shall have

$$a_x = \frac{N_x}{D_x}, \quad \mathbf{a}_x = 1 + a_x = \frac{\mathbb{N}_x}{D_x}.$$

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# CORRESPONDENCE.

## AN ASSURANCE FALLACY.

### *To the Editor of the Assurance Magazine.*

Sir,—The following problem presents several points of interest.

An assurance of A pounds is to be effected on $(x)$, at an annual premium $(\varpi)$, subject to the condition that interest on the premiums paid up to and including the year of death is to be allowed by the Office, at the rate involved in the tables employed, which rate it is assumed is that realized by the Office. Required $\varpi$.

Attempt a solution thus:— Since all the interest realized is to be handed over to $(x)$ or his representatives, the Office has obviously nothing but the bare premiums out of which to pay the sum assured. It is, therefore, as regards the Office, the same thing as if no interest were made; and we consequently need take account only of the average number of premiums that will be received from each policyholder. This number being $1 + e'_x$ (where $e'_x$ is the *curtate* mean duration of lives aged $x$), we have

$$\varpi(1 + e'_x) = A;$$

whence
$$\varpi = \frac{A}{1 + e'_x} \quad \ldots \ldots \ldots (1).$$

This is a very singular result. It is independent of the rate of interest; and yet it is obvious that the higher the rate realized by the Office the

greater will be the annual return to $(x)$, and consequently the less the cost of the assurance to him. The foregoing equation therefore cannot be true, and the process by which it is attained must be fallacious.*

But where, then, is the fallacy? It is in the assumption, tacitly made in the so-called solution, that the interest realized by the Office and that payable to the policyholders are identical. They are so, however, only as to *rate*, but not as to *amount*, except during the first year. At the end of that period the premium fund is so reduced by payment of death claims, that the interest yielded by it is no longer sufficient to meet that due to the policyholders. The deficiency, therefore, must be made good from the premiums themselves, and these therefore require to be increased to meet this charge.

The reasons why I have commenced with an erroneous solution, are— first, that an impression prevails, as I am informed, that this solution is a correct one; and secondly, that the problem belongs to a class which appear to invite the application of what are called common sense notions, while such applications usually lead, as in the present case, unless skilfully managed, to erroneous conclusions.

I now give a legitimate solution of the problem. The benefit consists of, first, a uniform assurance of $A$, the term corresponding to which is $AM_x$; and secondly, of an increasing annuity of $\varpi i$, $2\varpi i$, $3\varpi i$, &c., which makes its last payment at the end of the year of death. The term given by this annuity, *minus* its last payment, is $\varpi i S_x$, and that given by the last payment is $\varpi i R_x$. Hence, the payment term being $\varpi N_{x-1}$, we have

$$\varpi N_{x-1} = AM_x + \varpi i(S_x + R_x).$$

From this we obtain

$$\varpi = \frac{AM_x}{N_{x-1} - i(S_x + R_x)}.$$

Now,

$$N_{x-1} - i(S_x + R_x) = N_{x-1} - i(S_x + vS_{x-1} - S_x)$$

$$= N_{x-1} - (1-v)S_{x-1} = R_x.$$

$$\therefore \varpi = \frac{AM_x}{R_x} \quad \ldots \quad \ldots \quad (2).$$

Of the value of $\varpi$ thus determined it would be easy to show that for any value of $x$, except the oldest age in the table (for which $\varpi$ is always equal to $A$), it increases *with* (not *as*) $i$, the rate of interest.

Since, when $i$ diminishes without limit, $\dfrac{M_x}{R_x}$ approaches without limit to $\dfrac{D_x}{N_{x-1}}$ ; therefore, when $i=0$, *i.e.*, when money bears no interest, we have

$$\varpi = \frac{AD_x}{N_{x-1}} = \frac{A}{1+e'_x},$$

which agrees with (1). From this it appears that, although not true generally, (1) is true in the case of money bearing no interest. In this

* The reasoning here does not seem quite conclusive.—ED. *J. I. A.*

case, however, no interest being realized there is none payable to the policyholders.

The following table shows the premium per cent., by the Carlisle rate of mortality, at several rates of interest. The commutation table for $i = 0$ will be found at p. 145, vol. xiii. of the *Journal of the Institute of Actuaries.*

| Age. | $i = 0.$ | $i = \cdot03.$ | $i = \cdot04.$ | $i = \cdot05.$ |
|---|---|---|---|---|
| 30 | 2·8604 | 3·7290 | 4·1146 | 4·5707 |
| 50 | 4·6281 | 5·4515 | 5·7719 | 6·1156 |
| 70 | 10·3371 | 11·6040 | 12·0426 | 12·4872 |
| 90 | 26·4432 | 28·5903 | 29·3248 | 29·9864 |

For further elucidation of this somewhat curious problem I have worked out the following example at length, by the Carlisle table, at 5 per cent. The age is 90, and the sum assured £100. By (2) we get for the annual premium

$$\varpi = \frac{147\cdot9288}{4\cdot933192} = 29\cdot98643; \text{ whence } \varpi i = 1\cdot4993215.$$

```
142ϖ                          4253·0731                               1732·4009
5 per cent.                   *212·9037                               *86·6200
                              ─────────                               ─────────
                              4470·9768        5ϖi × 40   *299·8643   1819·0209
ϖi × 142    *212·9037         3912·9037        100 × 10   1000·       1299·8643
100 × 37    3700·             ─────────                               ─────────
                              558·0731         30ϖ                    519·1566
105ϖ                          3148·5752                               899·5929
                              ─────────                               
                              3706·6483                               1418·7495
                              *185·3324                               *70·9375
                              3891·9807        6ϖi × 30   *269·8779   1489·6870
2ϖi × 105   *314·8575         3314·8575        100 × 7    700·        969·8779
100 × 30    3000·             ─────────                               ─────────
                              577·1232         23ϖ                    519·8091
75ϖ                           2248·9822                               689·6879
                              ─────────                               
                              2826·1054                               1209·4970
                              *141·3053                               *60·4794
                              2967·4107        7ϖi × 23   *241·3908   1269·9719
3ϖi × 75    *337·3473         2437·3473        100 × 5    500·        741·3908
100 × 21    2100·             ─────────                               ─────────
                              530·0634         18ϖ                    528·5811
54ϖ                           1619·2672                               539·7557
                              ─────────                               
                              2149·3306                               1068·3368
                              *107·4665                               *53·4168
                              2256·7971        8ϖi × 18   *215·9023   1121·7536
4ϖi × 54    *323·8534         1723·8534        100 × 4    400·        615·9023
100 × 14    1400·             ─────────                               ─────────
                              532·9437         14ϖ                    505·8513
40ϖ                           1199·4572                               419·8100
                              ─────────                               ─────────
                              1732·4009                               925·6613
```

|  |  |  |  |  |  |
|---|---|---|---|---|---|
|  |  | 925·6613 |  |  | 514·7570 |
|  |  | *46·2830 |  |  | *25·7378 |
|  |  | 971·9443 |  |  | 540·4948 |
| 9ϖi × 14 | *188·9143 |  | 13ϖi × 5 | *97·4559 |  |
| 100 × 3 | 300· | 488·9145 | 100 × 2 | 200· | 297·4559 |
|  |  | 483·0308 |  |  | 243·0389 |
| 11ϖ |  | 329·8507 | 3ϖ |  | 89·9593 |
|  |  | 812·8815 |  |  | 332·9982 |
|  |  | *40·6440 |  |  | *16·6499 |
|  |  | 853·5255 |  |  | 349·6481 |
| 10ϖi × 11 | *164·9254 |  | 14ϖi × 3 | *62·9715 |  |
| 100 × 2 | 200· | 364·9254 | 100 × 2 | 200· | 262·9715 |
|  |  | 488·6001 |  |  | 86·6766 |
| 9ϖ |  | 269·8779 | ϖ |  | 29·9864 |
|  |  | 758·4780 |  |  | 116·6630 |
|  |  | *37·9239 |  |  | *5·8331 |
|  |  | 796·4019 |  |  | 122·4961 |
| 11ϖi × 9 | *148·4328 |  | 15ϖi × 1 | *22·4898 |  |
| 100 × 2 | 200· | 348·4328 | 100 × 1 | 100· | 122·4898 |
|  |  | 447·9691 |  |  |  |
| 7ϖ |  | 209·9050 |  |  |  |
|  |  | 657·8741 |  |  |  |
|  |  | *32·8937 |  |  |  |
|  |  | 690·7678 |  |  |  |
| 12ϖi × 7 | *125·9430 |  |  |  |  |
| 100 × 2 | 200· | 325·9430 |  |  |  |
|  |  | 364·8248 |  |  |  |
| 5ϖ |  | 149·9322 |  |  |  |
|  |  | 514·7570 |  |  |  |

Little explanation of the above is needed. At the outset the premium is received from the tabular number alive at 90, viz., 142, and a year's interest is added, giving a total in hand at the end of the first year of £4,470. This is immediately reduced by the payment of, first, £212·9037, interest on the premiums, and secondly, £3,700, the claims arising on 37 deaths, to £558·0731. The premium is again received from the 105 survivors, a year's interest is added, and the outgoings of the second year, amounting to £3314·8575, are deducted, leaving £577·1232 in hand at the commencement of the third year. In this way the scheme works itself out at the end of the fifteenth year.

It is visible now that after the first year the interest which the office realizes is altogether insufficient to meet that which it has to pay. And it is singular to note that, after the first few years, the ratio of the interest receivable (by the Office) to the interest payable, closely approximates to that of 1 : 4.* Whether this is accidental, or whether the like would be observed in other circumstances, I am at present unable to say.

Returning to equation (2), and writing it thus,

$$\varpi \mathrm{R}_x = \mathrm{AM}_x,$$

* To facilitate this comparison I have marked the interest on both sides with asterisks.

we see that the transaction resolves itself into an exchange or commutation of one assurance on $(x)$ for another, viz., a uniform assurance of A payable *by* the Office, and an increasing assurance of $\varpi$, $2\varpi$, &c. ($n\varpi$ in the $n$th year), payable *to* the Office. And this is correct, as it is obviously the same thing, theoretically, whether the premiums be paid annually, interest being allowed upon them, or in the aggregate at the end of the year of death. In practice, however, there is a great distinction between the two modes of payment. No Office would consent to defer the receipt of premium till the emergence of the claim, as they would in a great many cases have then more to receive than to pay.

It is interesting, however, to watch the operation of this mode of payment in a particular case; and I have therefore worked it out for the same age as before, 90, and at the same rate, 5 per cent. The premium also is of course the same, 29·98643.

| | | | | | |
|---|---|---|---|---|---|
| 100 × 37 | 3700· | | | | |
| $\varpi$ × 37 | 1109·4979 | 2590·5021 | | | 2852·6285 |
| 5 per cent. | | 129·5250 | | | 142·6314 |
| | | 2720·0271 | | | 2995·2599 |
| 100 × 30 | 3000· | | 100 × 3 | 300· | |
| $2\varpi$ × 30 | 1799·1858 | 1200·8142 | $9\varpi$ × 3 | 809·6336 | − 509·6336 |
| | | 3920·8413 | | | 2485·6263 |
| | | 196·0420 | | | 124·2813 |
| | | 4116·8833 | | | 2609·9076 |
| 100 × 21 | 2100· | | 100 × 2 | 200· | |
| $3\varpi$ × 21 | 1889·1451 | 210·8549 | $10\varpi$ × 2 | 599·7286 | − 399·7286 |
| | | 4327·7382 | | | 2210·1790 |
| | | 216·3869 | | | 110·5090 |
| | | 4544·1251 | | | 2320·6880 |
| 100 × 14 | 1400· | | 100 × 2 | 200· | |
| $4\varpi$ × 14 | 1679·2401 | − 279·2401 | $11\varpi$ × 2 | 659·7015 | − 459·7015 |
| | | 4264·8850 | | | 1860·9865 |
| | | 213·2442 | | | 93·0493 |
| | | 4478·1292 | | | 1954·0358 |
| 100 × 10 | 1000· | | 100 × 2 | 200· | |
| $5\varpi$ × 10 | 1499·3215 | − 499·3215 | $12\varpi$ × 2 | 719·6743 | − 519·6743 |
| | | 3978·8077 | | | 1434·3615 |
| | | 198·9404 | | | 71·7181 |
| | | 4177·7481 | | | 1506·0796 |
| 100 × 7 | 700· | | 100 × 2 | 200· | |
| $6\varpi$ × 7 | 1259·4301 | − 559·4301 | $13\varpi$ × 2 | 779·6472 | − 579·6472 |
| | | 3618·3180 | | | 926·4324 |
| | | 180·9159 | | | 46·3216 |
| | | 3799·2339 | | | 972·7540 |
| 100 × 5 | 500· | | 100 × 2 | 200· | |
| $7\varpi$ × 5 | 1049·5250 | − 549·5250 | $14\varpi$ × 2 | 839·6200 | − 639·6200 |
| | | 3249·7089 | | | 333·1340 |
| | | 162·4854 | | | 16·6567 |
| | | 3412·1943 | | | 349·7907 |
| 100 × 4 | 400· | | 100 × 1 | 100· | |
| $8\varpi$ × 4 | 959·5658 | − 559·5658 | $15\varpi$ × 1 | 449·7965 | − 349·7965 |
| | | 2852·6285 | | | |

The great distinction between this mode of arranging the transaction and the other is that there the Office was put in funds at the outset, enabling it to meet all claims as they arose, while here it is in advance from first to last.

If it is required to *load* the premium of this problem, we must proceed as in all cases in which the Office makes a return to the assured. It is not sufficient to apply the required loading to the value of $\varpi$, determined as above, since this would leave the additional interest which has to be returned unprovided for. The loading must, as in all such cases, be applied to the benefit side of the fundamental equation.

Let the required loading be $k$ per pound. Then,

$$\varpi N_{x-1} = (1+k)\{AM_x + \varpi i(S_x + R_x)\};$$

whence,

$$\varpi = \frac{(1+k)AM_x}{N_{x-1} - (1+k)i(S_x + R_x)}.$$

But, $N_{x-1} - (1+k)i(S_x + R_x) = N_{x-1} - (1+k)i(S_x + vS_{x-1} - S_x)$

$$= N_{x-1} - (1+k)(1-v)S_{x-1} = (1+k)\{N_{x-1} - (1-v)S_{x-1}\} - kN_{x-1}$$

$$= (1+k)R_x - kN_{x-1}.$$

$$\therefore \varpi = \frac{(1+k)AM_x}{(1+k)R_x - kN_{x-1}} = \frac{AM_x}{R_x - \dfrac{k}{1+k}N_{x-1}} \quad \dots \dots \quad (3).$$

This is obviously greater than $\dfrac{AM_x}{R_x}$; but it can be shown to be also greater than $\dfrac{(1+k)AM_x}{R_x}$, which is what the net premium becomes when the loading is *directly* applied to it. Thus,

$$\frac{AM_x}{R_x - \dfrac{k}{1+k}N_{x-1}} > (1+k)\frac{AM_x}{R_x},$$

if $\qquad\qquad R_x > (1+k)R_x - kN_{x-1},$

if $\qquad\qquad kN_{x-1} > kR_x,$

if $\qquad\qquad N_{x-1} > R_x;$

and this last we know to be true.

If no interest is earned, $M_x$ and $R_x$, as before, assume their limiting values, and (3) becomes

$$\varpi = \frac{AD_x}{N_{x-1} - \dfrac{k}{1+k}N_{x-1}} = (1+k)\frac{AD_x}{N_{x-1}}.$$

In this case, therefore, it suffices to apply the loading *directly* to the net premium; which is in accordance with the remark already made, the interest returnable by the Office being here *nil*.

I append a table of loaded premiums, corresponding to that already given of net premiums. The loading is 10 per cent., that is $k = \cdot 1$.

| Age. | $i=0.$ | $i=\cdot03.$ | $i=\cdot04.$ | $i=\cdot05.$ |
|---|---|---|---|---|
| 30 | 3·1464 | 4·5002 | 5·1881 | 6·1767 |
| 50 | 5·0909 | 6·3154 | 6·8386 | 7·4329 |
| 70 | 11·3708 | 13·0714 | 13·6844 | 14·3199 |
| 90 | 29·0875 | 31·8114 | 32·7722 | 33·6255 |

<div align="center">

I am, Sir,

Your most obedient servant,

P. GRAY.

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*London, 2nd Sept.,* 1867,

*\*\** A short note on the problem which forms the subject of this letter will be found in vol. v., p. 348.

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## VALUE OF A POLICY—FORMULÆ—MILNE.

### *To the Editor of the Assurance Magazine.*

DEAR SIR,—There is a theorem which I suppose must be in the heads of many actuaries, but I cannot find it in any of the books. It is that the values of a policy, as it runs on, are proportional to the falls in the value of the annuity. That is, if $a_x$ be the value of an annuity of £1 at the age $x$, the age of creation of the policy, the values of the policy at the ages $y$ and $z$ are as $a_x-a_y$ to $a_x-a_z$. That this theorem is not commonly expressed seems due to the value at the age $y$ being usually written

$$1-\frac{1+a_y}{1+a_x} \text{ instead of } \frac{a_x-a_y}{1+a_x}.$$

I shall be curious to see whether any one will produce a statement of this simple form. I find it occasionally very useful to take out from the table, without any writing, that the policy-value of $1+a_x$ at death is $a_x-a_y$ at the age $y$, the age $x$ being that of commencement. When a formula represents two different results, it is a useful exercise of ingenuity to deduce one result directly from the other. Now $a_x-a_y$ is the value to $(z)$ of a counter-survivorship—as we may call it—of the following kind. The executors of the first who dies pay an annuity of £1 to the survivor; and $(a_x-a_y)\div(1+a_z)$ is the whole-life premium which $(x)$ should pay to be put in this position. How, from the nature of this contract, does it follow that one payment of this premium, over and above the annual premium which $(x)$ should pay, admits $(y)$ to a policy of £1 at the premium for the age $(x)$?

Easy forms, corollaries from common forms, are things for *second editions.* A person who is engaged in a great effort, and has a heavy system of tables to look after, does not watch offshoots. Now none of the best known works—except only those of Price and Morgan, which lay no stress on formulæ—have arrived at second editions: this may be said of Baily, G. Davies, Milne, and David Jones.

It is much to be regretted that Milne did not, in his later years, occupy himself with a reconstruction of the algebraical part of his work. But it is hardly known how completely he abandoned the subject. In May,